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General Asymptotics of the Density of a Restricted Coalescing Random Walk System

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These results explore the asymptotic behavior of the density of a system of coalescing random walks where particles begin from only a subspace of the integer lattice and are allowed to walk anywhere on the lattice. They generalize results by Bramson and Griffeath from 1980.[1] Since the probability that a given site is occupied depends on how far that site is from the originating subspace, the density of the system at a given time must be re-defined. However, the general idea is still that if the density is larger than we expect at a given time, more coalescing events will occur, and the density will correct itself over time.

KEY WORDS: Coalescing random walk; hitting times; interacting particle system.

1. INTRODUCTION

In 1980, Bramson and Griffeath[1] found the specific rates at which the density of a system of coalescing random walks on the integer lattice $\mathbb{Z}^d$ decreases. This paper determines the rate at which the density of a restricted system of coalescing random walks decreases, where the particles do not begin from every site in $\mathscr{Z}^d$, but only from sites in a subspace of $\mathbb{Z}^d$.

A system of coalescing random walks on the lattice $\mathbb{Z}^d$ is a process consisting of particles which each start from a site in some subset D of $\mathbb{Z}^d$. Each particle behaves like an independent rate 1 simple symmetric random walk on $\mathscr{Z}^d$ until it jumps to a site already occupied by another particle. At this point the two particles coalesce into a single particle. Coalescing random walks are dual to the voter model. (See refs. 2, 4, and 6 for more information on this duality and interacting particle systems in general.) Because of this duality, the probability that the original opinion of the voter at the origin has survived until at least time t equals the probability

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that there is a particle at the origin at time t in the coalescing random walk system with $D=\mathbb{Z}^d$. This probability is the density of the coalescing random walk system. Let it be denoted by ϱ_t.

Bramson and Griffeath[1] used an important result by Sawyer[7] to find the specific rates at which the density ϱ_t decreases to zero: as $t \to \infty$,

$$\varrho_t \sim \begin{cases} \dfrac{1}{\sqrt{\pi t}} & \text{if} \quad d=1 \\ \dfrac{\log t}{\pi t} & \text{if} \quad d=2 \\ \dfrac{2G(0,0)}{t} & \text{if} \quad d>2, \end{cases}$$

where $G(0,0)$, the Green's function, is the expected amount of time a d-dimensional random walk spends at its starting point.

In contrast to ref. 1, this paper considers the behavior of the density of particles in a system of coalescing random walks with $D=H$, where H is an r-dimensional hyper-plane in $\mathbb{Z}^d$ (for $r<d$) parallel to the coordinate axes (this is for convenience—any r-dimensional subspace will do). In the case studied here, it is natural to consider the r-dimensional density of particles $p(t)$ in the system at time t. This paper defines $p(t)$ to be the limit of the expected number of particles in a large cube centered at the origin of side length N divided by N^r, the volume of the cube in $\mathbb{Z}^r$, as $N \to \infty$. Note that the usual d-dimensional density, such as used in ref. 1, is always zero. In contrast to the density defined in Bramson and Griffeath's work, $p(t)$ is *not* the probability that a given site is occupied at time t. In the system we consider, the probability that a given site is occupied at time t depends on how far that site is from H, the originating hyper-plane.

We first have the following elementary result.

Theorem 1.1. In the case $d-r>2$, there is a positive probability that a given random walk never coalesces with any other random walk.

The main result of this paper is the following.

Theorem 1.2. In the case $d-r=1$ or $d-r=2$, let

$$f(t) = \begin{cases} \dfrac{\sqrt{t}}{\log t} & \text{for} \quad d=2, \quad r=1 \\ \sqrt{t} & \text{for} \quad d-r=1 \quad \text{and} \quad d>2 \\ \log t & \text{for} \quad d-r=2. \end{cases}$$

Then there exist constants δ and β in $(0, \infty)$, independent of t, such that $\delta \leqslant p(t)\, f(t) \leqslant \beta$, for t larger than some t_0.

In this case, when $d > 2$, the probability that a $(d-r)$-dimensional random walk has not returned to its starting point by time t is on the order of $1/f(t)$. Also note that $f(t) \approx G_t(0, 0)$, where $G_t(0, 0)$ is the expected amount of time a $(d-r)$-dimensional random walk spends at its starting point until time t. We must make a correction in the case $d = 2$ because random walks are recurrent here. In the case $d = 2$, any site visited by the random walk will be visited on order of $\log t$ times in time t.

The proof of Bramson and Griffeath's result is based on the idea that if the density of the coalescing random walk system is larger than we expect at time t, the system will correct itself in an additional length of time, since the coalescing events will be more frequent than usual. The proof of Theorem 1.2 will also use this same general idea. However, the r-dimensional density $p(t)$ is not uniformly spread out over all of $\mathbb{Z}^d$. Yet, it will still be true that we can expect *most* of the particles to be near each other if the density is larger than we expect, and the coalescing dynamics will force the system down.

The rest of the paper is organized as follows. Section 2 contains the random walk estimates we will need. Section 3 contains the proof of Theorem 1.1. Section 4 contains the proof of the lower bound of Theorem 1.2. Sections 5 and 6 prove the upper bound in Theorem 1.2.

2. RANDOM WALK ESTIMATES

We shall see that the rate at which the density in our coalescing random walk system decreases hinges upon how likely it is that two random walks beginning a certain distance apart hit in a given time interval. Since it is difficult to directly analyze the behavior of particles in a system of coalescing random walks, we will compare the coalescing random walk system to systems of independent random walks.

We couple (in the obvious way) the system of *independent* rate one simple symmetric random walks $\{\zeta_s\}_{s \geqslant 0}$ and the system of *coalescing* rate one simple symmetric random walks $\{\xi_s\}_{s \geqslant 0}$. If the coalescing random walk ξ^0 hits ξ^i by time T, then the independent random walk ζ^0 has hit ζ^i by time T. Thus we have that the probability of the latter event is at least that of the former event. This simple inequality will allow us to use results about independent random walks for coalescing random walks when it is more convenient.

The first large deviation estimate that we will frequently use is a well-known estimate on the distance that an independent random walk travels in time t. (This result can be found on p. 29 in ref. 5 for discrete time random walks, and is easily extended to continuous time random walks.) For any $a>0$, there exists a finite constant c_a such that

$$P(|\zeta_t - \zeta_0| \geqslant n) \leqslant c_a e^{-an/\sqrt{t}} \tag{1}$$

for all $n, t>0$. The constant c_a is independent of n and t, but will depend on a. Since we can manipulate a and c_a, we will assume the distance n is measured in the ℓ_∞ norm. This bound is particularly useful since it allows us the flexibility to make the input of the exponential function as small as we need by taking large a and c_a.

The second estimate is a bound on the probability that a given random walk is at the origin at a given time. If ζ_t^x is the position of an independent rate one simple random walk at time t, then for any $a>0$,

$$P(\zeta_t^x = 0) \leqslant c_2 \left(\frac{c_1}{t^{d/2}} \wedge 1 \right) \exp\left\{ \frac{-a\,|x|}{\sqrt{t}} \right\}, \tag{2}$$

where c_1 is a constant depending only on dimension, and c_2 is a constant depending only on a. To prove this inequality, we first note that if the independent random walk ζ_t^x is at the origin at time t, then it has either traveled more than half the distance to the origin in the first $t/2$ units of time or in the last $t/2$ units of time. The result then follows from Eq. (1) and a weak version of the Local Central Limit Theorem (see refs. 3 and 8) which implies that the probability that $\zeta_{t/2}^w$ is at a given site z at time $t/2$ is uniformly bounded above in w and z by $c_1 t^{-d/2}$, for some constant c_1.

The last estimate we will need is a bound on the probability that two random walks in the coalescing system meet by time T, where T is the square of the original distance between them. We will use the next lemma to do this.

Lemma 2.1. Let τ_i be the hitting time of the independent rate one simple random walks ζ^0 and ζ^i for $i \in \mathbb{Z}^d$. Then we have the existence of constants c_1 and c_2 independent of i such that for $|i| \gg 1$,

$$c_1 < P(\tau_i \leqslant |i|^2) \log |i| < c_2 \qquad \text{for} \quad d=2, \qquad \text{and}$$

$$c_1 < P(\tau_i \leqslant |i|^2)\, |i|^{d-2} < c_2 \qquad \text{for} \quad d>2.$$

Proof. Since $\zeta^i - \zeta^0$ is a rate 2 random walk started from $i \in \mathscr{Z}^d \backslash \{0\}$, we relate $P(\tau_i \leqslant |i|^2)$ to the expected amount of time a rate two random walk begun at i spends at the origin until time $|i|^2 + |i|$. Let $\hat{\zeta}^i$ represent a rate 2 random walk started from $i \in \mathbb{Z}^d$, and $I^i(s)$ represent the expected amount of time $\hat{\zeta}^i$ spends at the origin until time s.

By the Strong Markov Property and Fubini's Theorem,[9]

$$\mathrm{E}\left[\int_0^{|i|^2+|i|} \mathbf{1}_{(\hat{\zeta}_s^i=0)}\, ds \,\middle|\, \tau_i \leqslant |i|^2\right] \geqslant \int_0^{|i|} \mathrm{P}(\hat{\zeta}_s^0=0)\, ds.$$

Thus

$$\mathrm{P}(\tau_i \leqslant |i|^2) \leqslant \frac{I^i(|i|^2+|i|)}{\int_0^{|i|} \mathrm{P}(\hat{\zeta}_s^0=0)\, ds}. \tag{3}$$

To find an upper bound for $I^i(|i|^2+|i|)$, first note that for $s=O(|i|^2)$, a weak version of the Local Central Limit Theorem (LCLT) again gives a constant c such that $\mathrm{P}(\hat{\zeta}_s^i=0) \leqslant cs^{-d/2}$. Thus there exists a constant c such that $\mathrm{P}(\hat{\zeta}_s^i=0) \leqslant c\,|i|^{-d}$. For $s=o(|i|^2)$, Eq. (2) implies that

$$\mathrm{P}(\hat{\zeta}_s^i=0) \leqslant \left(\frac{c}{s^{d/2}} \wedge 1\right) \exp\left\{\frac{-a\,|i|}{\sqrt{s}}\right\}.$$

As long as $|i| \gg 1$, these preceding two statements imply the existence of a constant c, not depending on i or s, such that $\mathrm{P}(\hat{\zeta}_s^i=0) \leqslant c\,|i|^{-d}$ for all $s<|i|^2+|i|$. Thus $I^i(|i|^2+|i|) \leqslant c\,|i|^{2-d}$ for some constant c independent of i.

The upper bound is now complete in the case $d>2$ once we simply note that the denominator of Eq. (3) is bounded below by a constant for $|i| \gg 1$.

For $d=2$, the LCLT implies the existence of another constant c such that $\mathrm{P}(\hat{\zeta}_s^0=0) \geqslant cs^{-1}$ for $s \gg 1$. This implies that as long as $|i| \gg 1$,

$$\int_0^{|i|} \mathrm{P}(\hat{\zeta}_s^0=0)\, ds \geqslant c \log |i|,$$

and the result follows.

Alternatively, we obtain a lower bound for the probability that τ_i is bounded above by $|i|^2$ by first using the Strong Markov Property to write

$$\mathrm{P}(\tau_i \leqslant |i|^2) \geqslant \frac{I^i(|i|^2)}{\int_0^{|i|^2} \mathrm{P}(\hat{\zeta}_s^0=0)\, ds}. \tag{4}$$

For the numerator, the LCLT implies that given fixed k, there exists a constant c such that $\mathrm{P}(\hat{\zeta}^i_s = 0) \geqslant c/|i|^d$ for $s \geqslant k\,|i|^2$. Thus we have another constant c such that

$$I^i(|i|^2) = \int_0^{|i|^2} \mathrm{P}(\hat{\zeta}^i_s = 0)\, ds \geqslant \frac{c}{|i|^{d-2}}.$$

Since random walks are transient in $d > 2$, we can now conclude $\mathrm{P}(\tau_i \leqslant |i|^2)$ is bounded below by $c\,|i|^{2-d}$ for some other constant c.

In the case $d = 2$, by Eq. (2) there exists a constant c such that

$$\int_0^{|i|^2} \mathrm{P}(\hat{\zeta}^0_s = 0)\, ds \leqslant \int_0^{|i|^2} \left(\frac{c}{s} \wedge 1\right) ds.$$

Thus as long as $|i| \gg 1$, the result follows. ∎

3. PROOF OF THEOREM 1.1

The basic idea of the proof is as follows. We handle sites that are near or far from the origin separately. The sum of probabilities of hitting walks started from sites far enough from the origin will be less than one. By conditioning on a well-chosen event A, we can still show that the sum of probabilities of hitting other walks will be less than one even after including the sites near the origin.

We create the distinction between the "near" and "far" sites by defining the event A as follows. Let B be the box in $\mathbb{Z}^d$ centered at the origin with side length N, where N is some large constant to be determined later. Fix a direction orthogonal to the hyperplane H from which the walks begin. Let A be the event that the coalescing random walk ξ^0 travels very far in this direction in time 1, and all other random walks started in B remain frozen during that time. Specifically, for some constant m to be determined later, let

$$\begin{aligned} A = \{&\xi^0_t \in \{\tilde{0}\} \times [0, m] \text{ for } \tilde{0} \in \mathbb{Z}^{d-1} \text{ and } t < 1\} \\ &\cap \{\xi^0_1 = (\tilde{0}, m)\} \\ &\cap \{\xi^x_t = \xi^x_0 \text{ for } x \in B \backslash \{0\} \text{ and } t \leqslant 1\}. \end{aligned}$$

Since A occurs with positive probability, we can prove the theorem by showing that given A, the probability that a given random walk avoids all other random walks is positive. Let σ_x be the hitting time of the coalescing random walks ξ^0 and ξ^x.

Fix a site $x \in H$ outside the box B. If the coalescing random walks ξ^x and ξ^0 coalesce before time 1, we can use Eq. (1). If they hit after time 1, we can use Lemma 2.1 (recall that since $d-r>2$, $d>2$). Thus for $x \in H \backslash B$,

$$P(\exists t,\ \xi_t^x = \xi_t^0 \mid A) \leqslant e^{-c\,|x|} + \frac{c}{|x|^{d-2}}. \tag{5}$$

For nonzero sites $x \in H \cap B$, the coalescing random walk ξ^x is frozen until time 1. Lemma 2.1 implies that

$$P(\exists t,\ \xi_t^x = \xi_t^0 \mid A) \leqslant P(\exists t,\ \zeta_t^x = \zeta_t^{(0,m)}) \leqslant \frac{c}{m^{d-2}}, \tag{6}$$

for some constant c.

Equations (5) and (6) imply that conditioned on A, the probability that the coalescing random walk started at zero ever hits any other particle is bounded above by

$$c\left[\sum_{x \in H \backslash B}\left\{e^{-|x|} + \frac{1}{|x|^{d-2}}\right\} + \sum_{x \in B \backslash \{0\}} \frac{1}{m^{d-2}}\right]. \tag{7}$$

Since the number of sites in H distance n from the origin is bounded above by cn^{r-1} for some constant c, Eq. (7) is bounded above by

$$c\sum_{n>N}\left\{n^{r-1}e^{-n} + \frac{n^{r-1}}{n^{d-2}}\right\} + \frac{(N^r-1)}{m^{d-2}}.$$

Since $d-r>2$, N can be chosen so that the first summand above is less than 1/3. We can then choose m so that the second summand is also less than 1/3. Therefore, the probability that ξ_t^0 avoids coalescing with any other random walk started on H is positive.

4. THEOREM 1.2: PROOF OF THE LOWER BOUND

Fix $t>0$. To show the density must be at least $\delta/f(t)$ for some δ independent of the fixed t, we compare the density at time t of our system of coalescing random walks $\{\xi_t\}_{t \geqslant 0}$ with the density at time t of a sparse coalescing random walk system $\{\eta_s\}_{s \geqslant 0}$ defined below. This sparse system will begin with random walks starting from sites far enough apart so that the initial density of the system is already on the order of $1/f(t)$, and that the probability that a given particle coalesces with any other particle by the fixed time t is as small as we need.

Recall that we defined

$$f(t)=\begin{cases}\dfrac{\sqrt{t}}{\log t} & \text{for} \quad d=2, \quad r=1\\ \sqrt{t} & \text{for} \quad d-r=1 \quad \text{and} \quad d>2\\ \log t & \text{for} \quad d-r=2.\end{cases} \tag{8}$$

For some large scaling index N that will be fixed later, let $\{\eta_s\}_{s\geqslant 0}$ consist only of particles begun from sites in H that are distance $Nf(t)^{1/r}$ apart from each other (recall t is fixed). Thus the initial r-dimensional density of the η-system is $c/f(t)$ where c is a constant depending only on N. By making N large enough, the second condition above will also be satisfied—if the chance of coalescing is small enough, the η-density will remain bounded below by something on the order of $f(t)^{-1}$.

Specifically, let $k=\lfloor f(t)^{1/r}\rfloor$ and let $\{\eta_s\}_{s\geqslant 0}$ be the system started from particles only at sites iNk for $i\in H$. Let $\rho(s)$ be the r-dimensional density of this system (defined similarly to $p(t)$), and let P_ρ be the probability law associated with this sparse system. Let η_s^x be the position at time s of the random walk started at x in the sparse system.

As noted above, the choice of N will be made from a bound on the probability that η^0 coalesces with any other particle by time t.

Lemma 4.1. The probability that a given particle in the sparse η-system coalesces with any other particle by the fixed time t is no more than cN^{1-d} where c is some constant independent of t.

Proof. For $d>2$, and $i\in H$ such that $|iNk|<k^{d-1}$, Lemma 2.1 implies

$$\mathrm{P}_\rho(\exists s\leqslant t:\eta_s^0=\eta_s^{iNk})\leqslant\frac{c}{|iNk|^{d-2}}.$$

For $|iNk|>k^{d-1}$, we find a nice bound by noting that the probability that two random walks hit by time t is the probability that they hit given that they come within radius $\sqrt{t}$ of each other times the probability they come within radius $\sqrt{t}$ of each other. The time they have to hit after coming within $\sqrt{t}$ of each other is random but is bounded by t, so we can use Lemma 2.1 to bound the former probability and Eq. (1) to bound the latter probability. Thus the probability that a given particle has coalesced with another by time t (the same t fixed above), is bounded above by

$$\sum_{|i| < (k^{d-2}/N)} \frac{c}{(|i|\, Nk)^{d-2}} + \sum_{|i| \geqslant (k^{d-2}/N)} 2 \exp\left\{\frac{-(|i|\, Nk - \sqrt{t})}{2\sqrt{t}}\right\} \frac{c}{\sqrt{t}^{\,d-2}}, \tag{9}$$

where $i \in \mathbb{Z}^r$ and c is some constant independent of t.

By making a change of variables, and by using the fact that $r-1 \leqslant d-2$, we get that

$$\mathrm{P}_\rho(\exists x \in H, s \leqslant t : \eta_s^0 = \eta_s^x) \leqslant \frac{c}{N^{d-1}} + \frac{c}{N^r},$$

where c is independent of t.

For $d=2$, the computation is similar but much simpler. In this case we conclude that

$$\mathrm{P}_\rho(\exists x \in H, s \leqslant t : \eta_s^0 = \eta_s^x) \leqslant \frac{c}{N},$$

where c is some constant independent of t. ∎

To conclude the proof of the lower bound in Theorem 1.2, we first note that $\rho(s)$, the density of the sparse system, is bounded below by $\rho(0)$ times the probability that a given particle has not coalesced in time s. By Lemma 4.1, one can choose N such that the probability that η^0 does *not* coalesce with any other particle is greater than 1/2. Since $\rho(0) = (1/Nk)^r$, we can now choose δ' such that $\rho(0) > \delta'/f(t)$. Thus since the η-system is dominated by our ξ-system,

$$p(t) \geqslant \rho(t) \geqslant \rho(0)\, \mathrm{P}_\rho(\forall x \in H : \eta_t^0 \neq \eta_t^x) > \frac{\delta'}{2f(t)}. \tag{10}$$

5. THEOREM 1.2: PROOF OF THE UPPER BOUND IN THE CASE THAT THE CO-DIMENSION IS 1

To obtain an upper bound on the density of particles as time increases, we show that if the density of particles is too large at a large time t, the coalescing dynamics will correct the system and force the density down in an additional amount of time t. However the coalescing dynamics can only affect particles “close” to each other. Some particles will be quite far from any other particle at time t and thus unlikely to hit any other particle in this additional time.

In particular, we expect independent rate 1 random walks to travel about distance $\sqrt{s}$ in time s. Thus, to differentiate particles that have moved "too far" from H in time s, we define

$$L(s) = \mathscr{Z}^r \times [-M\sqrt{s}, M\sqrt{s}]^{d-r},$$

for some large integer M to be fixed later. $L(s)$ is a slab surrounding H—by making M large, we control the proportion of particles moving outside $L(s)$ by time s (and thus away from the bulk of the other random walks in the system). This is the key difference between $\{\xi_s\}$ and the system studied in Bramson and Griffeath's paper.[1] In a process where random walks are begun from *every* site in $\mathscr{Z}^d$, it makes no difference whether or not a particle makes a large deviation—it will always be surrounded by other particles since the random walks are evenly distributed across all of $\mathscr{Z}^d$.

Since some of the particles in $\{\xi_s\}$ will stray far from any other, we split particles at time $2t$ into three main categories:

1. those outside $L(t)$ at time t,

2. those in $L(t)$ at time t, but that have traveled outside $L(2t)$ at some point at or before time $2t$,

3. those in $L(t)$ at time t, and that have remained in $L(2t)$ through time $2t$.

To show that the density of particles that have strayed far from H is as small as we need in the case $d-r=1$, we will need to introduce a score function for particles to emphasize the large distance such particles have traveled. Each particle that is outside of $L(s)$ at time s will have a score associated with it: a particle at $(0, y)$ for $0 \in \mathscr{Z}^r$ and $y \in \mathscr{Z}^{d-r}$ will be assigned the score $e^{|y|/\sqrt{s}}$. Notice that the score of such a particle will be at least e^M. Since it is unlikely that a particle will continue to remain outside $L(s)$ as s increases, a high score will be forced down as the ratio of position to the square root of time decreases.

To take advantage of the score function, we will not study $p(s)$, the density at time s, directly. Since an equivalent definition of $p(s)$ is the expected number of particles in $\{\tilde{0}\} \times \mathscr{Z}^{d-r}$, where $\tilde{0} \in \mathscr{Z}^r$, we can bound $p(s)$ from above by

$$q(s) = \mathrm{E}\left[\sum_{|y| > M\sqrt{s}} \mathbf{1}_{\{\exists x \in H : \xi_s^x = (0, y)\}} e^{|y|/\sqrt{s}}\right] + \mathrm{E}\left[\sum_{|y| \leqslant M\sqrt{s}} \mathbf{1}_{\{\exists x \in H : \xi_s^x = (0, y)\}}\right]. \tag{11}$$

We shall call this the revised density of particles. Let $q_1(t)$ be the first summand and $q_2(t)$ be the second. Our ultimate goal is to show that if the revised density of particles is much larger than we expect at time t, then in an additional t units of time, it will be forced down. Note that $q(2t)$ is the sum of the revised densities of the three categories.

Since the probability of leaving $L(s)$ by time s is exponentially small in M, the next two lemmas will establish that the density of the particles in the first two classes will be quite small even if we ignore the coalescing dynamics, as long as M is chosen large enough.

Lemma 5.1. At time $2t$, the revised density of particles in category 1 is bounded above by $\beta_1 q(t)$, where $\beta_1 = c_1 e^{-c_2 M}$, for some constants c_1 and c_2 independent of t and M.

Proof. Using Eq. (11), the revised density of category 1 particles at time $2t$ is written as a sum indexed over the position of the particles at time $2t$. Since we are just looking for an upper bound, we can actually ignore the fact that some particles in category 1 will have died in the time interval $[t, 2t]$. Thus for the purposes of this proof, we can re-index the sum using the position of the particles at time t instead. By the Markov Property, the density of category 1 particles at time $2t$ can be bounded above by

$$\sum_{|y| > M\sqrt{t}} \mathrm{E}\mathbf{1}_{\{\exists x \in H : \xi_t^x = (0, y)\}} \exp\left\{\frac{|y|}{\sqrt{2t}}\right\} \mathrm{E} \exp\left\{\frac{|\xi_t^0|}{\sqrt{2t}}\right\}. \tag{12}$$

The expectation term in Eq. (12) is in fact independent of t. To see this, first note that if $|\xi_t^0|/\sqrt{2t}$ is in the annulus $\{n \leqslant x < n+1\}$, then ξ_t^0 has moved at least distance n in time t. Thus

$$\mathrm{E} \exp\left\{\frac{|\xi_t^0|}{\sqrt{2t}}\right\} \leqslant \sum_{n=0}^{\infty} \exp\{n+1\}\ \mathrm{P}(|\xi_t^0| \geqslant n\sqrt{2t}). \tag{13}$$

By Eq. (1), there exists a constant c_a such that $\mathrm{P}(|\xi_t^0| \geqslant n\sqrt{2t})$ is bounded above by $c_a \exp\{-a\sqrt{2n}\}$ for any choice of a. The power of Eq. (1) is clear: we can choose any a such that this sum converges. Therefore Eqs. (12) and (13) imply that the revised density of particles in category 1 at time $2t$ is bounded above by

$$c \sum_{|y| > M\sqrt{t}} \mathrm{E}\mathbf{1}_{\{\exists x \in H : \xi_t^x = (0, y)\}} \exp\left\{\frac{|y|}{\sqrt{2t}}\right\} \leqslant c e^{-M/4} q_1(t) < c e^{-M/4} q(t). \quad \blacksquare$$

Lemma 5.2. The revised density of particles in category 2 at time $2t$ is bounded above by $\beta_2 q(t)$, where $\beta_2 = c_1 e^{-c_2 M}$, for some constants c_1, c_2 independent of M and t.

Proof. Suppose a category 2 particle has traveled outside $L(2t)$ by time $2t$ to a site $(0, y)$, for $0 \in \mathbb{Z}^{d-1}$ and $y \in \mathbb{Z}$ such that $|y| > M\sqrt{2t}$, and suppose $(0, y)$ is the farthest such point to which it travels by time $2t$. Let $r \in (t, 2t]$ be the time at which it (first) hits this maximum. Thus this random walk has traveled at least distance $|y| - M\sqrt{t}$ in time $r - t$. Using Eq. (1), the probability of this event is bounded above by

$$c_1 \exp\left\{-4\,\frac{|y| - M\sqrt{t}}{\sqrt{r-t}}\right\}$$

for some constant c_1. (Again we use the flexibility of choosing any a in Eq. (1).) Such a particle will also gain a score at time $2t$ of at most $\exp\{|y|/\sqrt{r}\}$.

A little algebra will show that there exists a positive finite constant c_2 such that

$$-4\left(\frac{|y| - M\sqrt{t}}{\sqrt{r-t}}\right) + \frac{|y|}{\sqrt{r}} \leqslant -c_2 M. \tag{14}$$

This inequality thus implies that the revised density of particles that were in $L(t)$ at time t but move outside $L(2t)$ at some point before time $2t$ is bounded above by

$$c_1 e^{-c_2 M} q_2(t) \leqslant c_1 e^{-c_2 M} q(t). \quad \blacksquare$$

What remains is to consider the density of category 3 particles—those that were in $L(t)$ at time t and never leave $L(2t)$ through time $2t$. For convenience, call this class $\mathcal{N}$ and its density at time s, $q_{\mathcal{N}}(s)$. We need to show that if $q_{\mathcal{N}}(s)$ is too large for some time s, $t \leqslant s \leqslant 2t$, then the coalescing dynamics will force it down. We start by partitioning $[t, 2t]$ into pieces of time length $t_\varepsilon^{\,2}$, where

$$t_\varepsilon = \begin{cases} \varepsilon\sqrt{\dfrac{t}{\log t}} & \text{if} \quad d = 2 \\ \varepsilon t^{1/d} & \text{if} \quad d > 2 \end{cases} \tag{15}$$

for a small constant ε chosen later. Since particles that are at least distance t_ε away from any other particle at time s are unlikely to coalesce by time $s + t_\varepsilon^{\,2}$, we will call these particles *isolated* at time s.

The density of isolated particles is small simply because only a limited density of boxes with volume $t_\varepsilon{}^d$ can fit in $L(2t)$—thus the following lemma is easily proved.

Lemma 5.3. Let $q_i(s)$ be the density of particles in class $\mathcal{N}$ that are isolated at time s for $t \leqslant s \leqslant 2t$. Let α_0 be chosen so that $cM/\varepsilon^d \alpha_0 < 1/2$. Then if the density of class $\mathcal{N}$ particles at time s is more than $\alpha_0/f(s)$, $q_i(s) < q_{\mathcal{N}}(s)/2$.

The final building block for the lower bound is as follows

Lemma 5.4. For α_0 as chosen as in Lemma 5.3, either $q_{\mathcal{N}}(2t)\, f(2t) \leqslant \alpha_0$ or $q_{\mathcal{N}}(2t) \leqslant q_{\mathcal{N}}(t)/8$.

Proof. Recall that if we let

$$g(\omega) = \begin{cases} \log \omega & \text{if} \quad d = 2 \\ \omega^{d-2} & \text{if} \quad d > 2, \end{cases}$$

Lemma 2.1 allows us to choose a constant γ independent of t_ε, x, and y such that the probability that two particles at most distance t_ε apart at time s coalesce by time $s + t_\varepsilon{}^2$ is is bounded below by $\gamma/g(t_\varepsilon)$.

For $s \in [t, 2t]$ let $q_c(s+u)$, $u \geqslant 0$, be the density of particles remaining at time $s+u$ that are in class $\mathcal{N}$ but not isolated at time s. Then

$$q_c(s + t_\varepsilon{}^2) \leqslant q_c(s) \left(1 - \frac{\gamma}{2g(t_\varepsilon)}\right). \tag{16}$$

By Lemma 5.3, if $q_{\mathcal{N}}(s)\, f(s) > \alpha_0$, then $q_i(s) < q_{\mathcal{N}}(s)/2$, and thus $q_c(s) \geqslant q_{\mathcal{N}}(s)/2$. Therefore Eq. (16) implies

$$q_{\mathcal{N}}(s + t_\varepsilon{}^2) \leqslant q_i(s) + q_c(s) \left(1 - \frac{\gamma}{2g(t_\varepsilon)}\right) \leqslant q_{\mathcal{N}}(s) \left(1 - \frac{\gamma'}{g(t_\varepsilon)}\right), \tag{17}$$

where $\gamma' = \gamma/4$, as long as $q_{\mathcal{N}}(s)\, f(s) > \alpha_0$.

What Eq. (17) provides us with is a "bootstrap" algorithm that allows us to step in time increments of length $t_\varepsilon{}^2$ from time t to $2t$ as long as $q_{\mathcal{N}}(t + jt_\varepsilon{}^2)\, f(t + jt_\varepsilon{}^2) > \alpha_0$ for $j = 0, 1,..., t/t_\varepsilon{}^2$. Thus we can conclude that either $q_{\mathcal{N}}(2t)\, f(2t) \leqslant \alpha_0$ or that

$$q_{\mathcal{N}}(2t) \leqslant q_{\mathcal{N}}(t) \left(1 - \frac{\gamma'}{g(t_\varepsilon)}\right)^{t/t_\varepsilon^2} \leqslant q_{\mathcal{N}}(t) \exp\left\{\frac{-\gamma' t}{t_\varepsilon{}^2 g(t_\varepsilon)}\right\}. \tag{18}$$

To finish the proof, one can choose ε such that the exponential term in Eq. (18) is less than 1/8. ∎

So far we have shown the following: at time $2t$ the revised densities of category 1 and 2 particles are very small with respect to $q(t)$; and, if the density of category 3 particles is too large at time t, then by time $2t$, this density will be reduced. Two questions remain: how much do these results reduce $q(2t)$, and is the resulting decrease in the total revised density enough to conclude Theorem 1.2?

To answer the first question, we know from Lemmas 5.1 and 5.2 that $q(2t) \leqslant (\beta_1 + \beta_2)\, q(t) + q_{\mathcal{N}}(2t)$, and further that we can choose M large enough that $\beta_1 + \beta_2 < 1/8$. Therefore since $f(2t) \leqslant \sqrt{2}\, f(t)$ for any $d \geqslant 2$,

$$q(2t)\, f(2t) \leqslant \left(\frac{q(t)}{8} + q_{\mathcal{N}}(2t) \right) \sqrt{2}\, f(t). \tag{19}$$

Since M and ε have finally been chosen, we can choose α_0 as in Lemma 5.3. We now see that if $q_{\mathcal{N}}(2t)$ is larger than the combined densities of the other two categories of particles and it is larger than $\alpha_0 / f(t)$ then the fact that it is forced down implies the entire density at time $2t$ has been forced down. More specifically, in the case that $q_{\mathcal{N}}(2t) > q(t)/8$, and $q_{\mathcal{N}}(2t)\, f(2t) \geqslant \alpha_0$, Lemma 5.4 implies $q_{\mathcal{N}}(2t) \leqslant q_{\mathcal{N}}(t)/8 \leqslant q(t)/8$. Thus Eq. (19) implies $q(2t)\, f(2t) \leqslant q(t)\, f(t)/2$.

If instead, $q_{\mathcal{N}}(2t)$ is larger than $q(t)/8$ but still below the threshold level ($q_{\mathcal{N}}(2t)\, f(2t) \leqslant \alpha_0$), then Eq. (19) implies $q(2t)\, f(2t) \leqslant 2\alpha_0$.

Finally if $q_{\mathcal{N}}(2t)$ is smaller than the combined densities of the other two categories, Eq. (19) immediately implies that $q(2t)\, f(2t) \leqslant q(t)\, f(t)/2$.

To answer the second question, let

$$\beta = \max_{1 \leqslant t \leqslant t_0} \{ q(t)\, f(t),\, 4\alpha_0 \}.$$

Suppose at some time $t \geqslant t_0$, $q(t)\, f(t) > \beta$. But then for any n such that $t/2^n < t_0$,

$$q\left(\frac{t}{2^n}\right) f\left(\frac{t}{2^n}\right) > 2^n \beta,$$

and we have a contradiction of our definition of β.

6. THEOREM 1.2: PROOF OF THE UPPER BOUND IN THE CASE THAT THE CO-DIMENSION IS 2

The proof in this case is in the same spirit as the one for $d - r = 1$. From the lower bound we have seen that the density is decreasing at a much slower rate. Thus to see that the system will correct itself if the density is larger than we expect at time t, we need to wait until time t^2, not $2t$. At any time $s > t\log^2 t$ we again split the existing particles into classes:

1. those outside $L(s)$,
2. those inside $L(s)$ but "isolated" (this term will be redefined), and
3. those inside $L(s)$ and not isolated.

These categories are similar to those in Section 5, but there are important differences. Categories 1 and 2 from Section 5 are lumped together into Category 1 here. We can do this because we can do away with the score function and directly show that for any time $s > t\log^2 t$, the density of particles outside $L(s)$ at time s is very small in comparison to $p(t)$. A further difference in these classes is that in Section 5 we were able to treat isolated and non-isolated particles in $L(s)$ together. In this case, we must treat them separately since t_ε cannot be constant over the entire time interval $[t, t^2]$.

As before, we treat each category in turn with a similar sequence of lemmas. Lemma 6.1 is analogous to Lemmas 5.1 and 5.2. Lemma 6.2 is analogous to Lemma 5.3. Finally Lemma 6.3 is analogous to Lemma 5.4.

We begin with category 1.

Lemma 6.1. Let $p_o(s)$ be the density of particles outside $L(s)$ at time s. For t larger than some t_0, and for $s > t\log^2 t$, $p_o(s)$ is bounded above by $\beta_1 p(t)$, where $\beta_1 = c_1 e^{-c_2 M}$.

Proof. Suppose that ξ_s is a particle that is outside of $L(s)$ at time s. Then either $|\xi_s - \xi_t| \geqslant M\sqrt{s}/2$ or $|\xi_t| \geqslant M\sqrt{s}/2$.

In the first case, the event $|\xi_s - \xi_t| \geqslant M\sqrt{s}/2$ is independent of the event that ξ_t is alive at time t by the Markov Property. Thus Eq. (1) and the Markov Property imply that the density of particles that fall in this case is bounded above by

$$cp(t)\exp\left\{\frac{-M\sqrt{s}}{2\sqrt{s-t}}\right\} \leqslant c\exp\left\{\frac{-M}{2}\right\} p(t)$$

for some constant c.

In the second case, the density of particles ξ_t such that $|\xi_t| \geqslant M\sqrt{s}/2$ is bounded above by the density of such particles in the independent system $\{\zeta_t\}$. Thus Eq. (1) implies this density is bounded above by

$$\mathrm{P}\left(\frac{|\zeta_t|}{\sqrt{s}} \geqslant \frac{M}{2}\right) \leqslant \exp\left\{\frac{-M\log t}{2}\right\}, \tag{20}$$

as long as $s > t\log^2 t$.

Since Eq. (20) is bounded above by ct^{-1}, and since Eq. (10) implies $p(t) \gg t^{-1}$, there exists a t_0 such that for $t \geqslant t_0$, this density is bounded above by $\exp\{-M/4\}\ p(t)$, for $s > t \log^2 t$. ∎

We now consider the particles that are in $L(s)$ at time s. As in the case $d-r=1$, we want to call a particle isolated at time s if it is at least distance $t_\varepsilon(s)$ away from any other. Here we would like $t_\varepsilon(s)$ to be $\varepsilon(s \log s)^{1/d}$ to ensure that the density of these particles in $L(s)$ is like $c/\log s$, for some small constant c. But this strategy presents a few bookkeeping problems. Instead we will divide the interval of time $[t \log^2 t, t^2]$ into disjoint regions where $t_\varepsilon(s)$ is constant. For each $r = 0,..., \log_2(t/\log^2 t) - 1$, let $T_r = 2^r t \log^2 t$. For $s \in [T_r, T_{r+1})$, let

$$t_\varepsilon(s) = \varepsilon(T_{r+1} \log T_{r+1})^{1/d}, \tag{21}$$

where ε is some small constant to be chosen later. Define a particle to be *isolated* at time s if there is no other particle within $t_\varepsilon(s)$ away. Again, simply by volume restrictions, we have the following lemma.

Lemma 6.2. Let $p_i(s)$ be the density of particles in $L(s)$ at time s that are isolated at time s for $s \in [T_r, T_{r+1})$. Let α_0 be a constant such that $cM^2/\varepsilon^d \alpha_0 < 1/4$. If $p(s)\ f(s) > \alpha_0$, then $p_i(s) < p(s)/4$.

The last lemma is our bootstrap lemma for co-dimension $d-r=2$.

Lemma 6.3. For α_0 chosen as in Lemma 6.2, either

$$p(t^2)\ f(t^2) \leqslant \alpha_0 \qquad \text{or} \qquad p(t^2)\ f(t^2) < p(t)\ f(t)/2$$

for t greater than some t_0.

Proof. Let $p_c(s)$ be the density of particles in category 3 for $s \in [T_r, T_{r+1})$. By a similar argument as in the proof of Lemma 5.4, if $p_c(s) \geqslant p(s)/2$, then $p_o(s) + p_i(s) \leqslant p(s)/2$, and Lemma 2.1 implies

$$p(s + t_\varepsilon^{\,2}) \leqslant p(s) \left(1 - \frac{\gamma'}{t_\varepsilon^{\,d-2}}\right)$$

for all $s \in [T_r, T_{r+1})$, where $\gamma' = \gamma/4$.

Again as argued in the proof of Lemma 5.4, we can then show that either $p(T_{r+1})\ f(T_{r+1}) \leqslant \alpha_0$, $p(t^2)\ f(t^2) \leqslant p(t)\ f(t)/2$, or that

$$p(T_{r+1}) \leqslant p(T_r) \exp\left\{\frac{-\gamma'}{4\varepsilon^d \log t}\right\}. \tag{22}$$

From Eq. (22), we can conclude that either $p(t^2)\, f(t^2) \leqslant \alpha_0$, $p(t^2)\, f(t^2) \leqslant p(t)\, f(t)/2$, or that

$$p(t^2) \leqslant p(T_0) \exp\left\{\frac{-\gamma' \log(t/\log^2 t)}{4\varepsilon^d \log t}\right\} \leqslant p(t) \exp\left\{\frac{-\gamma'}{8\varepsilon^d}\right\} \tag{23}$$

for t greater than some t_0. If we choose ε such that $\exp\{-\gamma'/8\varepsilon^d\} \leqslant 1/4$, we have that either $p(t^2)\, f(t^2) \leqslant \alpha_0$ or that

$$p(t^2)\, f(t^2) \leqslant \frac{p(t)}{4} \log t^2 \leqslant \frac{p(t)\, f(t)}{2}. \quad \blacksquare$$

By a similar argument as in the case where $d - r = 1$, this decrease in the overall density gives us the upper bound.

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